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Solution of the Fan Diagram Equation in $2+1$ Dimensional QCD

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Abstract

We investigate the Balitsky-Kovchegov (BK) equation for $D = 3$ space-time dimensions, corresponding to one transverse coordinate, and we show that it can be solved analytically. The explicit solutions are found in the linear approximation and for the paricular cases when they do not depend on the impact parameter of the dipole or on its rapidity. It is shown, that in a general case different solutions are related by an infinite parameter group of transformations. The key observation is that the equation has the Kowalewskaya-Painlevé property, which gives the possibility of reducing the non-linear problem to the solution of a linear integral equation.

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1 Introduction

In the framework of perturbative QCD the most general basis for the theoretical description of small-x processes is given by the BFKL approach [1], based on the gluon Reggeization. Taking into account also analyticity and unitarity constraints for the S-matrix one can calculate the high energy asymptotics of scattering amplitudes with a good accuracy [2]. In particular, using this approach the kernel of the BFKL equation was obtained in the next-to-leading order (NLO) [3]. To avoid a strong renormalization scale dependence of its solution in the $\overline{MS}$-scheme one can chose the Brodsky-Lepage-Mackenzie (BLM) optimal scale setting within physical renormalization schemes [4]. It is important that the BLM procedure allows to preserve, approximately, some good properties of the leading order approximation, in particular, its conformal invariance [5]. Incorporation of the corrections related to the renormalization group symmetry [6] should make the predictions of the BFKL equation for the small-x physics more reliable.

However, at asymptotically large energies the BFKL approach should be generalized, since it leads to a power-like growth of cross sections. The problem of the Froissart bound violation cannot be solved by the calculation of radiative corrections at any fixed order and requires other methods. One of the possibilities is to take into account the multi-reggeon contributions in the framework of the BKP equations [7]. A general framework for considering unitarization effects is the reggeon field theory, describing the interactions of reggeized gluons. A possible way of deriving such a field theory is based on the high-energy effective QCD action [8], generalizing the BFKL approach for multi-particle production and multi-Reggeon interactions.

In a first step, the unitarization program can be realized by non-linear generalizations of the BFKL equation, related to the idea of saturation of parton densities [9]. The most important ingredient is the transition vertex for $2 \to 4$ reggeized gluons which, in the BFKL approach, has been calculated in [10] and further investigated in [11, 12]. Using this transition vertex, as a first step beyond the BFKL equation, one can formulate a nonlinear evolution equation which sums the QCD fan diagrams consisting of BFKL Green's functions [13]. In a future step towards finding solutions to the reggeon field theory, closed loops of BFKL Pomerons should be included and calculated.

At present it is fashionable to describe the saturation in DIS within the color dipole approach [14], which in the target rest frame has an attractive and clear physical interpretation: the incoming γ^* splits into a $q\bar{q}$ colour dipole long before the dipole interacts with

the target. In this approach the cross section is presented as

$$\sigma_{\gamma^*}(x,Q^2) = \int d^2r \int_0^1 dz |\Psi_{\gamma^*}(r,z,Q^2)|^2 \sigma_{dp}(r,x), \tag{1}$$

where $\Psi_{\gamma^*}(r,z,Q^2)$ is the photon wave function, z the quark momentum fraction, and $r = |\vec{r}|$ the transverse size of the colour dipole, $\vec{r} = \vec{r}_1 - \vec{r}_2$; the transverse vectors $\vec{r}_1$ and $\vec{r}_2$ are the quark and antiquark coordinates, and $\sigma_{dp}(r,x)$ is the dipole cross section,

$$\sigma_{dp}(r,x) = 2\int d^2b \, (1 - S(\vec{r}_1,\vec{r}_2;Y)) \ . \tag{2}$$

Here $\vec{b} = (\vec{r}_1 + \vec{r}_2)/2$ is the impact parameter of the quark-antiquark system, $Y = \log(1/x)$, and $S(\vec{r}_1,\vec{r}_2;Y) = \langle tr\,(U^+(\vec{r}_1)U(\vec{r}_2))\,/N_c\rangle_Y$ is the S-matrix for the elastic scattering , which can be presented as an average of the product of two Wilson lines
$U^+(x_\perp) = P\exp[ig\int dx^- A_a^+(x^-,x_\perp)t^a]$.

There exist two different approaches for calculating the Y dependence of $S(\vec{r}_1,\vec{r}_2;Y)$. The first one starts from an infinite hierarchy of coupled equations [15] for products of any given number of Wilson lines. Taking the limit of large N_c and considering the case of the target being a large nucleus, the equation for the two-line correlator decouples and takes a simple form [16]:

$$\begin{aligned}\frac{\partial S(\vec{r}_1,\vec{r}_2,Y)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{r}\frac{(\vec{r}_1-\vec{r}_2)^2}{(\vec{r}_1-\vec{r})^2(\vec{r}_2-\vec{r})^2}\,&[S(\vec{r}_1,\vec{r},Y)S(\vec{r},\vec{r}_2,Y)\\ &-S(\vec{r}_1,\vec{r}_2,Y)] \ .\end{aligned} \tag{3}$$

At small $N = 1 - S$, one can neglect terms nonlinear in N, and this equation turns into the colour-dipole version of the BFKL equation for the dipole cross section N. In [13] it has been shown that Eq.(3) can be considered as a special limit of the QCD fan diagram equation mentioned before: apart from taking the large-N_c limit, the dipole cross section has to be in the Möbius class of functions. An alternative approach for describing saturation is the Color Glass Condensate [17]. In this framework, the Y-evolution is based upon functional equations. It is affirmed [18] that both approaches give identical results for the calculation of observables.

Eq.(3) has attracted much interest. First, numerical solutions have been found for a b-independent approximation [19], and later on also for the full b-dependent equation [20]. Several attempts have recently been made to find analytical solutions, at least in some approximation [21]. However, so far no exact analytical solution is known. Therefore it will be interesting to find a model in which the BK equation could be solved.

In this paper we generalize the BK-equation (3) to an arbitrary dimension D of the space-time. For the case $D = 2 + 1$ - which corresponds to one transverse dimension - we shall present analytical solutions (the BFKL dynamics was investigated at $D = 2 + 1$ in the papers [22]). The key observation in finding these solutions is that the equation has the Kowalewskaya-Painlevé property. We proceed in several steps of approximations. After writing down the equation in one transverse dimension (Section 2) we first discuss the linear approximation (Section 3). Then we turn to the nonlinear equation, beginning with the b-independent case (Section 4), where we find explicit solutions for arbitrary initial conditions. Passing on to the b-dependence we first find all "stationary" (rapidity independent) solutions (Section 5), generalizing then to the rapidity dependent case (Section 6). We conclude with the investigation of the infinite parameter symmetry of the equation (Section 7) and its general solution (Section 8). A few details are put into the Appendix.

2 The fan equation for 2+1 QCD

Let us begin by writing down the fan equation for the S-matrix describing the dipole evolution in the $(2 + 1)$-dimensional QCD. The kernel of the BK equation (3) is equal to the probability density in the space of transverse coordinates and rapidity for the soft gluon emission by the quark-antiquark pair:

$$\frac{\alpha_s N_c}{2\pi^2}\frac{(\vec{r}_1-\vec{r}_2)^2}{(\vec{r}_1-\vec{r})^2(\vec{r}_2-\vec{r})^2} = 2\alpha_s N_c \mid \int \frac{d^{D-2}k}{(2\pi)^{D-2}}\frac{\vec{k}}{\vec{k}^2}(e^{i\vec{k}(\vec{r}_1-\vec{r})}-e^{i\vec{k}(\vec{r}_2-\vec{r})}) \mid^2 \tag{4}$$

at $D = 4$. It is clear that if the space-time dimension takes the value $D = 4 + 2\epsilon$, the BK equation preserves its form, but since

$$\int \frac{d^{2+2\epsilon}k}{(2\pi)^{2+2\epsilon}}\frac{\exp{(i\vec{k}\vec{r})}\vec{k}}{\vec{k}^2} = \frac{i}{2\pi}\frac{\Gamma(1+\epsilon)}{\pi^\epsilon}\left(\frac{\vec{r}}{(\vec{r}^2)^{1+\epsilon}}\right), \tag{5}$$

the kernel changes according to

$$\frac{\alpha_s N_c}{2\pi^2}\frac{(\vec{r}_1-\vec{r}_2)^2}{(\vec{r}_1-\vec{r})^2(\vec{r}_2-\vec{r})^2} \longrightarrow \frac{\alpha_s N_c}{2\pi^2}\left(\frac{\Gamma(1+\epsilon)}{\pi^\epsilon}\right)^2\left(\frac{(\vec{r}_1-\vec{r})}{(\vec{r}_1-\vec{r})^{2(1+\epsilon)}}-\frac{(\vec{r}_2-\vec{r})}{(\vec{r}_2-\vec{r})^{2(1+\epsilon)}}\right)^2 . \tag{6}$$

Putting $\epsilon = -1/2$ we obtain:

$$\frac{\partial S_{\rho_2\rho_1}(y)}{\partial y} = \int_{\rho_1}^{\rho_2} d\rho_0\,(S_{\rho_2\rho_0}(y)S_{\rho_0\rho_1}(y) - S_{\rho_2\rho_1}(y))\ ,\ y = \frac{\alpha_s}{2\pi}Y\ , \tag{7}$$

where the inequality $\rho_2 > \rho_1$ is imposed without any restriction of generality. Note that the scattering amplitude $N_{\rho_2\rho_1}(y)$ is related to the S-matrix $S_{\rho_2\rho_1}(y)$ by the identity

$$N_{\rho_2\rho_1}(y) = 1 - S_{\rho_2\rho_1}(y)\,. \tag{8}$$

We introduce the dipole impact parameter b and its size ρ

$$b = \frac{\rho_2 + \rho_1}{2}\,, \;\; \rho = \rho_2 - \rho_1\,, \;\; \rho' = \rho_0 - \rho_1\,. \tag{9}$$

Then the fan equation can be presented as follows

$$\frac{\partial S(b,\rho,y)}{\partial y} = \int_0^\rho d\rho' \left(S(b+\frac{\rho'}{2},\rho-\rho',y)\, S(b-\frac{\rho-\rho'}{2},\rho',y) - S(b,\rho,y)\right)\,, \tag{10}$$

or, in a simpler way,

$$\left(\frac{\partial}{\partial y} + \rho\right)\, S(b,\rho,y) = \int_0^\rho d\rho' S(b-\frac{\rho'}{2},\rho-\rho',y)\, S(b+\frac{\rho-\rho'}{2},\rho',y)\,. \tag{11}$$

Now we go to the mixed representation (b,p)

$$S(b,\rho,y)\,\theta(\rho) = \int_{-i\infty}^{i\infty} \frac{dp}{2\pi i}\, e^{\rho p}\, s(b,p,y) \tag{12}$$

with the Laplace transform

$$s(b,p,y) \; = \int_0^\infty d\rho\, e^{-\rho p}\, S(b,\rho,y). \tag{13}$$

In this representation ρ is the differential operator

$$\rho\, s(b,p,y) = -\frac{\partial}{\partial p}\, s(b,p,y) \equiv -\partial s(b,p,y)\,. \tag{14}$$

The convolution of two functions in the ρ-representation

$$F_{12}(\rho) = \int_0^\rho d\rho' F_1(\rho-\rho')\; F_2(\rho') \tag{15}$$

is reduced to the product of their Mellin transforms

$$f_{12}(p) = f_1(p)\; f_2(p)\,, \;\; f(p) \; = \int_0^\infty d\rho\, e^{-\rho p}\, F(\rho)\,. \tag{16}$$

Moreover, a more general relation holds

$$f_{12}^{kl}(p) = \left(\partial^l f_1(p)\right)\; \left(\partial^k f_2(p)\right) \tag{17}$$

for the Mellin transform of the function

$$F_{12}^{kl}(\rho) = \int_0^\rho d\rho'\, (-\rho')^k (-\rho+\rho')^l F_1(\rho-\rho')\, F_2(\rho')\,. \tag{18}$$

Let us introduce the anti-normal ordering $\overline{N}$ of operators depending on the two momenta p_r $(r=1,2)$ and the corresponding derivatives $\partial_r = \partial/\partial(p_r)$, according to the rules

$$\overline{N}(p_r\partial_s) = \overline{N}(\partial_s p_r) \equiv \partial_s p_r = p_r\partial_s + \delta_{rs}\,. \tag{19}$$

Then one can verify that the fan equation in the mixed representation (b,p) can be written in the form

$$\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial p}\right) s(b,p,y)=\lim_{p_r\to p}\overline{N}\left(s(b+\frac{1}{2}\partial_2,p_1,y)\,s(b-\frac{1}{2}\partial_1,p_2,y)\right)1\,, \tag{20}$$

where it is implied that, after performing the transformation of the differential operator to the anti-normal form, one omits all terms with the differential operators due to the relations

$$\partial_1^l\partial_2^k 1=0\,,\;\;k,l\geq 1$$

and, at the end, puts $p_1=p_2=p$.

Note that one can consider a modified equation for the S-matrix in the case of a non-zero temperature T in the t-channel (see [23]). In this case $S(\rho,y)$ is assumed to be a periodic function of ρ_k

$$S_{\rho_2\rho_1}(y)=S_{\rho_2\rho_1}(y)_{|\rho_k\to\rho_k+1/T}\,. \tag{21}$$

3 The BFKL equation in $2+1$ dimensional QCD

To begin with, we consider the linearized case corresponding to the BFKL equation in $2+1$ dimensional space-time. Using the relation $S(b,\rho,y)=1-N(b,\rho,y)$ and neglecting nonlinear contributions in (11), one can obtain

$$\left(\frac{\partial}{\partial y}+\rho\right)N(b,\rho,y)=\int_0^\rho d\rho'\left(N(b-\frac{\rho'}{2},\rho-\rho',y)+N(b+\frac{\rho'}{2},\rho-\rho',y)\right)\,. \tag{22}$$

This equation is simplified significantly in the case when N does not depend on b

$$\left(\frac{\partial}{\partial y}+\rho\right)N(\rho,y)=2\int_0^\rho d\rho' N(\rho',y)\,. \tag{23}$$

For the Mellin transform

$$n(p,y)\;=\int_0^\infty d\rho\,e^{-\rho p}\,N(\rho,y)\,,\quad N(\rho,y)\,\theta(\rho)=\int_{-i\infty}^{i\infty}\frac{dp}{2\pi i}\,e^{\rho p}\,n(p,y) \tag{24}$$

we obtain the differential equation

$$(\frac{\partial}{\partial y}-\frac{\partial}{\partial p})\,n(p,y)=\frac{2}{p}\,n(p,y)\,. \tag{25}$$

Its solution with the initial condition $n(p,0)=n_0(p)$ is

$$n(p,y)=\frac{(y+p)^2\,n_0(y+p)}{p^2}\,. \tag{26}$$

The second order pole at $p=0$ provides a linear dependence of $N(\rho,y)$ at large ρ

$$N(\rho,y) \simeq \rho\, y^2\, n_0(y)\,, \tag{27}$$

which leads to a violation of the S-matrix unitarity at $y \neq 0$ for any initial condition. Note that Eq.(23) has a stationary (y-independent) solution $N(\rho,y) = c\rho$, with arbitrary c. But this solution is not acceptable from the physical point of view.

Let us turn now to Eq.(22) for the amplitude depending on b. Performing the Mellin and Fourier transformations in the variables ρ and b, respectively,

$$N(b,\rho,y)\,\theta(\rho) = \int_{-i\infty}^{i\infty} \frac{dp}{2\pi i}\, e^{\rho p} \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, e^{iqb}\, n(q,p,y)\,,$$

$$n(q,p,y) \;=\; \int_0^{\infty} d\rho\, e^{-\rho p} \int_{-\infty}^{\infty} db\, e^{-ibq}\, N(b,\rho,y),) \tag{28}$$

we obtain

$$(\frac{\partial}{\partial y} - \frac{\partial}{\partial p})\, n(q,p,y) = \frac{8p}{4p^2+q^2} n(q,p,y)\,. \tag{29}$$

Its general solution is

$$n(q,p,y) = \frac{f(p+y;q)}{4p^2+q^2}\,, \tag{30}$$

with the arbitrary function $f(p+y;q)$. For the initial condition $n(q,p,0) = n_0(q,p)$ the result can be written as follows

$$n(q,p,y) = \frac{(4(y+p)^2+q^2)\, n_0(q,y+p)}{4p^2+q^2}\,, \tag{31}$$

where

$$n_0(q,p) = \int_0^{\infty} d\rho\, e^{-\rho p} \int_{-\infty}^{\infty} db\, e^{-ibq}\, N(b,\rho,0)\,. \tag{32}$$

Therefore

$$\begin{aligned} N(b,\rho,y) \;&=\; \int_{-i\infty}^{i\infty} \frac{dp}{2\pi i}\, e^{\rho p} \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, e^{iqb}\, \frac{(4(y+p)^2+q^2)}{4p^2+q^2} \\ &\times \int_0^{\infty} d\rho'\, e^{-\rho'(p+y)} \int_{-\infty}^{\infty} db'\, e^{-ib'q}\, N(b',\rho',0)\,. \end{aligned} \tag{33}$$

By introducing the Green function $G_y(b,\rho;b',\rho',0)$ according to the definition

$$N(b,\rho,y) = \int_0^{\infty} d\rho'\, e^{-\rho' y} \int_{-\infty}^{\infty} db'\, G_y(b,\rho;b',\rho')\, N(b',\rho',0)\,, \tag{34}$$

we obtain for it

$$\begin{aligned} G_y(b,\rho;b',\rho') \;&= \delta(b-b')\delta(\rho-\rho') \\ +\theta(\rho-\rho') \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, e^{iq(b-b')} &\left(2y\cos\frac{q(\rho-\rho')}{2} + \frac{2y^2}{q}\sin\frac{q(\rho-\rho')}{2}\right)\,, \end{aligned} \tag{35}$$

so that the Green function for the BFKL Pomeron is a polynomial in y

$$G_y(b,\rho;b',\rho') \,=\delta(b-b')\,\delta(\rho-\rho')$$
$$+\theta(\rho-\rho')\sum_{\lambda=\pm}\left(y\,\delta(b-b'+\lambda\frac{\rho-\rho'}{2})+\lambda y^2\theta(b-b'+\lambda\frac{\rho-\rho'}{2})\right). \tag{36}$$

Therefore the y-evolution of the initial dipole distribution $N_{\rho_2\rho_1}(0)$ in the (ρ_2,ρ_1)-representation is simple:

$$N_{\rho_2\rho_1}(y)=e^{-y\rho_{21}}N_{\rho_2\rho_1}(0)+y\int_{\rho_1}^{\rho_2}d\rho_0\left(e^{-y\rho_{20}}N_{\rho_2\rho_0}(0)+e^{-y\rho_{01}}N_{\rho_0\rho_1}(0)\right)$$
$$+y^2\int_{\rho_1}^{\rho_2}d\rho_2'\int_{\rho_1}^{\rho_2'}d\rho_1' e^{-(\rho_2'-\rho_1')y}N_{\rho_2'\rho_1'}(0). \tag{37}$$

For large $y\gg 1/\rho$, where in (30) one can neglect p in comparison with y, the solution in the coordinate representation is

$$N(b,\rho,y)\,=\int_{-\infty}^{\infty}\frac{dq}{2\pi}\,e^{iqb}\,\frac{\sin(\rho q/2)}{2q}(4y^2+q^2)\,n_0(q,y)\,. \tag{38}$$

In particular, for large $b\gg\rho$ we obtain the above discussed expression $N(b,\rho,y)\simeq\rho\,y^2\,n_0(y)$. On the other hand, at $b\ll\rho$ one has

$$N(b,\rho,y)\,\simeq\int_{-\infty}^{\infty}\frac{dq}{2\pi}\,\frac{\sin(\rho q/2)}{q}\,2y^2\,n_0(q,y)\,. \tag{39}$$

Note, that Eq. (29) admits stationary solutions

$$n(q,p,y)\,=\frac{f(q)}{4p^2+q^2} \tag{40}$$

with an arbitrary function $f(q)$. For $N(b,\rho,y)$ we obtain correspondingly

$$N(b,\rho,y)\,=a(b+\frac{\rho}{2})-a(b-\frac{\rho}{2}) \tag{41}$$

with an arbitrary amplitude a. Actually, the existence of such solutions can be seen from Eq. (22). They are given by a difference of the amplitudes depending on transverse coordinates of quark ρ_1 and antiquark ρ_2, respectively. Splitting of the dipole does not lead to any evolution of such a solution: new quarks and antiquarks on the edges of the dipole have the same transverse coordinates, and therefore their contributions cancel in the linear approximation.

4 Solutions depending only on the dipole size

Next let us investigate the simple case when the solution does not depend on b, which, formally, corresponds to the asymptotics of $s(b,p,y)$ at $b \to \infty$

$$\lim_{b\to\infty} s(b,p,y) = s(p,y). \tag{42}$$

This function describes the S-matrix for the scattering of the dipole with the size ρ on the hadron system with a large radius $R \gg \rho$, provided that the impact parameter b is restricted to the interval

$$R \gg b \gg \rho\ .$$

In this case the fan equation is significantly simplified. In terms of variables

$$\xi = \frac{1}{2}(y-p)\,, \quad \eta = \frac{1}{2}(y+p) \tag{43}$$

it takes the form:

$$\frac{\partial}{\partial \xi}\, s(\xi,\eta) = s^2(\xi,\eta)\,. \tag{44}$$

In accordance with the fact that this is a degenerated Riccati equation, this equation has the Kowalewskaya-Painleve property [24], and its solution is a simple meromorphic function

$$s(\xi,\eta) = \frac{1}{\xi(\eta)-\xi}\,. \tag{45}$$

Returning to the ρ-representation, we obtain the following result

$$S(\rho,y) = \int_{-i\infty}^{i\infty} \frac{dp}{2\pi i}\, e^{\rho p}\, \frac{1}{\left(s(p+y)\right)^{-1} - y}\,, \tag{46}$$

where $s(p)$ is the Laplace transform of the $S-$matrix at the initial moment $y=0$

$$s(p) = \int_0^\infty d\rho\, e^{-\rho p}\, S(\rho,0)\,. \tag{47}$$

Assuming the colour transparency property in the form

$$\lim_{\rho\to 0} S(\rho,0) = 1 - c\rho^\gamma + \ldots\,,\ \ \gamma > 0\,,\, c > 0\,, \tag{48}$$

one can calculate the function $s(p,y)$ at large y or p

$$s(p,y) \simeq \frac{1}{p + c\,\Gamma(1+\gamma)\,(p+y)^{1-\gamma}}\ . \tag{49}$$

Therefore, if $0 < \gamma < 1$, we obtain in the region $p \sim y^{1-\gamma} \ll y$ the scaling behaviour

$$s(p,y) \simeq \frac{1}{p + c\,\Gamma(1+\gamma)\, y^{1-\gamma}}\,, \tag{50}$$

which leads to the S-matrix

$$S(\rho, y) \simeq e^{-c\Gamma(1+\gamma)\, y^{1-\gamma}\rho} \tag{51}$$

in the region $y^{1-\gamma}\rho \sim 1$. It means, that at large y the distribution in ρ is proportional to a smeared δ-function $\delta(\rho)$.

If, on the other hand, $\gamma > 1$ (for example, when $\lim_{\rho\to 0} S(\rho, 0) = 1 - c\rho^2$) the S-matrix tends to unity in the large region $\rho \ll y^{\gamma-1}$, which corresponds to the vanishing of the dipole interaction. In the particular case when $\gamma = 1$ and

$$\lim_{\rho\to 0} S(\rho, 0) = 1 - c\rho + \ldots\,, \quad c > 0\,, \tag{52}$$

the asymptotic behaviour of the S-matrix at large y is especially simple:

$$\lim_{y\to\infty} S(\rho, y) = e^{-c\rho}\,. \tag{53}$$

It corresponds to the fact that in the 2 + 1-dimensional QCD the fan equation has the stationary solutions

$$S(\rho, y) = e^{-c\rho}\,, \tag{54}$$

corresponding to the conservation of the total length of dipoles. This expression is similar to the Maxwell distribution in energies for molecules in a thermal equilibrium. In our case the energy is proportional to the total length ρ of dipoles. Thus, there is an analogy between the Boltzman and BK equations.

5 Stationary solutions of the fan equation

The stationary fan equation

$$0 = \int_{\rho_1}^{\rho_2} d\rho_0\, (S_{\rho_2\rho_0}\, S_{\rho_0\rho_1} - S_{\rho_2\rho_1}) \tag{55}$$

has the following solution

$$\widetilde{S}_{\rho_2\rho_1} = \frac{S(\rho_2)}{S(\rho_1)}\,, \tag{56}$$

where $S(\rho)$ is an arbitrary function. In a linear approximation it corresponds to the amplitude $N(b, \rho, y)$ (41) for the BFKL equation, being the sum of two functions depending on ρ_1 and ρ_2. In order to show that $\widetilde{S}_{\rho_2\rho_1}$ is a general solution, we expand the equation in terms of small fluctuations $\Delta S_{\rho_2\rho_1}$ around the solution

$$\Delta S_{\rho_2\rho_1} = S_{\rho_2\rho_1} - \widetilde{S}_{\rho_2\rho_1}\,, \tag{57}$$

and we obtain, in the linear approximation,

$$0 = \int_{\rho_1}^{\rho_2} d\rho_0 \left(\widetilde{S}_{\rho_2\rho_0}\, \Delta S_{\rho_0\rho_1} + \Delta S_{\rho_2\rho_0}\, \widetilde{S}_{\rho_0\rho_1} - \Delta S_{\rho_2\rho_1} \right) . \tag{58}$$

By introducing the relative correction $\widetilde{\delta S}_{\rho_2\rho_1}$

$$\Delta S_{\rho_2\rho_1} = \widetilde{S}_{\rho_2\rho_1} \widetilde{\delta S}_{\rho_2\rho_1} \,, \tag{59}$$

we simplify the above equation

$$0 = \int_{\rho_1}^{\rho_2} d\rho_0 \left(\widetilde{\delta S}_{\rho_0\rho_1} + \widetilde{\delta S}_{\rho_2\rho_0} - \widetilde{\delta S}_{\rho_2\rho_1} \right) . \tag{60}$$

There is an obvious solution of this equation

$$\widetilde{\delta S}_{\rho_2\rho_1} = s(\rho_2) - s(\rho_1), \tag{61}$$

where $s(\rho)$ is an arbitrary function. This correction, however, represents only a redefinition of the function $S(\rho)$ in the expression for $\widetilde{S}_{\rho_2\rho_1}$. In fact, by introducing the new variables $\rho = \rho_2 - \rho_1$, $b = (\rho_2 + \rho_1)/2$ we can reduce the above equation for $\delta S_{\rho_2\rho_1}$ to the stationary equation for $N(b, \rho, y)$, which has only the solutions $\widetilde{\delta S}_{\rho_2\rho_1}$, as it was shown in Section 3.

The stationary fan equation has the Kowalewskaya-Painleve property. Namely, if we the solution in the series

$$S_{\rho_2\rho_1} = \sum_{n=0}^{\infty} a_n(b)\, \rho^n; \quad a_0 = 1, \tag{62}$$

the first coefficient $a_1(b)$ is an arbitrary function of ρ with the following asymptotic behaviour at $b \to \infty$

$$a_1(b) = \sum_{r=1}^{\infty} \frac{c_r}{b^r} \,. \tag{63}$$

Indeed, the above solution of the stationary equation can be written as follows

$$\widetilde{S}_{\rho_2\rho_1} = \prod_{k=1}^{\infty} \left(\frac{1 + \frac{\rho}{2(b+d_k)}}{1 - \frac{\rho}{2(b+d_k)}} \right)^{r_k} \,, \tag{64}$$

where d_k and r_k are arbitrary complex parameters. Therefore, the above function $a_1(b)$ is

$$a_1(b) = \sum_{k=1}^{\infty} \frac{r_k}{b + d_k} \,. \tag{65}$$

In contrast to this, as a consequence of the Kowalewskaya-Penleve property, for a given $a_1(b)$ the solution $\widetilde{S}_{\rho_2\rho_1}$ is obtained in an unique way. For example, if we start from the scaling ansatz

$$a_1(b) = \frac{a}{b} \,, \tag{66}$$

the stationary solution is

$$\widetilde{S}_{\rho_2\rho_1} = \left(\frac{b+\frac{\rho}{2}}{b-\frac{\rho}{2}}\right)^a \simeq 1 + \, a\,\frac{\rho}{b} + \frac{a^2}{2}\frac{\rho^2}{b^2} + \left(\frac{a}{12}+\frac{a^3}{6}\right)\frac{\rho^3}{b^3} + O\left(\rho^4\right)\;. \tag{67}$$

We can also write solutions of the stationary equation which do not have any singularities at real $\rho_{1,2}$. For example,

$$\widetilde{S}_{\rho_2\rho_1} = \frac{1+\rho_1^2}{1+\rho_2^2}\,. \tag{68}$$

6 Scaling solution of the non-stationary equation

Let us now consider the general case when $s(b,p,y)$ depends on b, and let us introduce the new variables

$$\eta = \frac{1}{2}(y+p)\,,\;\; \epsilon = \xi(\eta) - \frac{1}{2}(y-p)\,, \tag{69}$$

where $\xi(\eta)$ is an arbitrary function which later on will be derived from the requirement that $s(b,p,y)$ satisfies initial conditions at $y=0$. In terms of the new variables the fan equation looks as follows

$$-\frac{\partial}{\partial\epsilon}\,S(b,\epsilon,\eta) = \lim_{\eta_r\to\eta,\epsilon_r\to\epsilon}\overline{N}$$
$$\left(S(b+\frac{1}{4}\left(\frac{\partial}{\partial\eta_2}+(1+\xi'(\eta_2))\,\frac{\partial}{\partial\epsilon_2}\right),\epsilon_1,\eta_1)\,S(b-\frac{1}{4}\left(\frac{\partial}{\partial\eta_1}+(1+\xi'(\eta_1))\,\frac{\partial}{\partial\epsilon_1}\right),\epsilon_2,\eta_2)\right)1\,. \tag{70}$$

Its solution can be searched in the form of an expansion in powers of $1/\epsilon$:

$$\widetilde{S}(b,\epsilon,\eta) = \frac{1}{\epsilon} + \sum_{r=1}^{\infty}\Delta_r\widetilde{S}(b,\xi,\eta)\,,\;\Delta_r\widetilde{S}(b,\xi,\eta) = \frac{c_r(b,\eta)}{\epsilon^{r+1}}\,. \tag{71}$$

Here $c_1(b,\eta)$ turns out to be an arbitrary function, fixed by initial conditions for $s(b,p,y)$ at $y=0$. According to the fan equation the residues $c_r(b,\eta)$, near $b=\infty$, are analytic functions with the following asymptotic behaviour:

$$c_r(b,\eta) = \frac{1}{b^r}\sum_{s=0}^{\infty}\frac{c_{r,s}(\eta)}{b^s}\,. \tag{72}$$

In particular, for the first correction $\Delta_1 S(b,\epsilon,\eta)$ to the pole $1/\epsilon$ we obtain the differential equation

$$\left(-\frac{\partial}{\partial\epsilon}-\frac{2}{\epsilon}\right)\Delta_1 S(b,\epsilon,\eta) = 0\,. \tag{73}$$

Its solution is

$$\Delta_1 S(b,\epsilon,\eta) = \frac{c_1(b,\eta)}{\epsilon^2}\,, \tag{74}$$

where

$$c_1(b,\eta) = \frac{1}{b}\sum_{s=0}^{\infty}\frac{c_{1,s}(\eta)}{b^s} \tag{75}$$

is an arbitrary function which is analytic and vanishing at $b\to\infty$. It should be chosen to satisfy the initial conditions for $s(b,\rho,y)$ at $y=0$.

Therefore, in the expansion of $S(b,\epsilon,\eta)$ in powers of $1/\epsilon$, there is a phenomenon similar to the Kowalewskaya-Painleve property for the integrable differential equations. Namely, one can construct the general solution in the class of meromorphic functions by fixing the functions $\xi(\eta)$ and $c_1(b,\eta)$ from the initial conditions for $s(b,p,y)$. The next correction $\Delta_2 S(b,\epsilon,\eta)$ satisfies the equation

$$\left(-\frac{\partial}{\partial\epsilon}-\frac{2}{\epsilon}\right)\Delta_2\, S(b,\epsilon,\eta) =$$

$$\frac{1}{\epsilon^4}\left(\overline{N}\, c_1(b+\frac{1}{4}\frac{\partial}{\partial\eta_2},\eta_1).c_1(b-\frac{1}{4}\frac{\partial}{\partial\eta_1},\eta_2)_{|\eta_2=\eta_1=\eta} - \frac{1}{2}\frac{\partial}{\partial b}c_1(b+\frac{1}{4}\frac{\partial}{\partial\eta_2},\eta)\,(1+\xi'(\eta_2))_{|\eta_2=\eta}\right), \tag{76}$$

where, in the last term, it is implied that after the differentiation of c_1 the removed operator $b+\frac{1}{4}\frac{\partial}{\partial\eta_2}$ is substituted by the expression $1+\xi'(\eta_2)$ in the corresponding place. Therefore we obtain

$$\Delta_2 S(b,\epsilon,\eta) = \frac{c_2(b,\eta)}{\epsilon^3}\,, \tag{77}$$

where

$$c_2(b,\eta) = \overline{N}\, c_1(b+\frac{1}{4}\frac{\partial}{\partial\eta_2},\eta_1).c_1(b-\frac{1}{4}\frac{\partial}{\partial\eta_1},\eta_2)_{|\eta_2=\eta_1=\eta} - \frac{1}{2}\frac{\partial}{\partial b}c_1(b+\frac{1}{4}\frac{\partial}{\partial\eta_2},\eta)\,(1+\xi'(\eta_2))_{|\eta_2=\eta}\,. \tag{78}$$

In an analogous way one can calculate the residues $c_r(b,\eta)$ in terms of $\xi(\eta)$ and $c_k(b,\eta)$ with $k<r$.

Let us now consider the solution of the fan equation in the scaling regime

$$\epsilon \ll 1\,,\ b\gg 1\,,\ \epsilon b\sim 1, \tag{79}$$

for fixed η. In this case the equation (70) is significantly simplified

$$-\frac{\partial}{\partial\epsilon}\,S(b,\epsilon) = \lim_{\epsilon_r\to\epsilon}\overline{N}\,\left(S(b+u\frac{\partial}{\partial\epsilon_2},\epsilon_1)\,S(b-u\frac{\partial}{\partial\epsilon_1},\epsilon_2)\right)1\,, \tag{80}$$

where the parameter $u=u(\eta)$ is defined by the expression

$$u=\frac{1}{4}\left(1+\xi'(\eta)\right)\,, \tag{81}$$

and the solution has the following expansion at large ϵb

$$S(b,\epsilon) = \frac{1}{\epsilon} + \frac{v}{\epsilon^2 b} + \dots \,. \tag{82}$$

Here the function $v = v(\eta)$ is fixed by the initial conditions for $s(b,\, p,\, y)$. The first two terms on the rhs of (82) determine all the other coefficients of the expansion

$$S(b,\epsilon) = \frac{1}{\epsilon}\sum_{n=0}^{\infty}\frac{a_n}{(\epsilon b)^n}; \quad a_0 = 1, \;\; a_1 = v. \tag{83}$$

The reccurence relations for the coefficients a_n are given in the Appendix, and one obtains

$$S(b,\epsilon) = \frac{1}{\epsilon} + \frac{v}{\epsilon^2 b} + \frac{v^2}{\epsilon^3 b^2} + \left(2u^2 v + v^3\right)\frac{1}{\epsilon^4 b^3} + \dots \tag{84}$$

It is easy to see that the substitution

$$S(b,\epsilon) \to \frac{1}{2u} s(b, \frac{\epsilon}{2u})\,, \quad \epsilon \to 2up \tag{85}$$

turns (80) into (20) for the stationary (y-independent) case. Therefore we have the remarkable result: the general solution of the non-stationary equation at $\epsilon b \sim 1$ can be expressed in terms of the stationary solution depending on ρ/b. Using the stationary solution obtained in the previous section (see (67)) one can derive

$$S(b,\epsilon) = \frac{1}{2u}\int_0^\infty d\rho\, e^{-\frac{\epsilon\rho}{2u}}\left(\frac{b+\frac{\rho}{2}}{b-\frac{\rho}{2}}\right)^{\frac{v}{2u}} \,. \tag{86}$$

7 Infinite parameter symmetry of the fan equation

In this section we consider another ansatz:

$$S_{\rho_2\rho_1}(y) = \widetilde{S}_{\rho_2\rho_1}(y) = e^{(\chi(\rho_2)-\chi(\rho_1))y}\,\frac{R(\rho_2)}{T(\rho_1)}\,, \tag{87}$$

where $R(\rho)$ and $T(\rho)$ are arbitrary functions. Putting this ansatz into Eq.(7) it is easy to verify that it is indeed a non-stationary solution, provided that

$$\chi(\rho) = \int_0^\rho d\rho_0\left(\frac{R(\rho_0)}{T(\rho_0)} - 1\right) \,. \tag{88}$$

For small fluctuations $\delta N_{\rho_2\rho_1}(y)$ around this solution

$$S_{\rho_2\rho_1}(y) \simeq \widetilde{S}_{\rho_2\rho_1}(y)(1 + \delta N_{\rho_2\rho_1}(y)) \tag{89}$$

we obtain, inserting (89) into (7) and linearizing the result in $\delta N_{\rho_2\rho_1}(y)$),

$$\frac{\partial \delta N_{\rho_2\rho_1}(y)}{\partial y} = \int_{\rho_1}^{\rho_2} d\rho_0 \frac{R(\rho_0)}{T(\rho_0)} \left(\delta N_{\rho_2\rho_0}(y) + \delta N_{\rho_0\rho_1}(y) - \delta N_{\rho_2\rho_1}(y)\right) . \tag{90}$$

We have "zero mode" solutions of this equation

$$\widetilde{\delta N}_{\rho_2\rho_1}(y) = \delta R(\rho_2) - \delta T(\rho_1) + y\left(\delta\chi(\rho_2) - \delta\chi(\rho_1)\right) , \tag{91}$$

where $\delta R(\rho)$ and $\delta T(\rho)$ are arbitrary functions related to $\delta\chi(\rho)$ as follows

$$\delta\chi(\rho) = \int_0^{\rho} d\rho_0 \frac{R(\rho_0)}{T(\rho_0)} \left(\delta R(\rho_0) - \delta T(\rho_0)\right) . \tag{92}$$

Introducing the new integration variable $\Upsilon(\rho)$

$$\Upsilon(\rho) = \int^{\rho} d\rho_0 \frac{R(\rho_0)}{T(\rho_0)}, \tag{93}$$

we obtain the following linear equation for $\delta N_{\rho_2\rho_1}(y) \equiv N_{\Upsilon_2\Upsilon_1}(y)$:

$$\frac{\partial N_{\Upsilon_2\Upsilon_1}(y)}{\partial y} = \int_{\Upsilon_1}^{\Upsilon_2} d\Upsilon_0 \, \left(N_{\Upsilon_2\Upsilon_0}(y) + N_{\Upsilon_0\Upsilon_1}(y) - N_{\Upsilon_2\Upsilon_1}(y)\right) , \tag{94}$$

which coincides with the BFKL equation having the general solution given by (33) with the substitution

$$\rho = \rho_{21} \to \Upsilon_2 - \Upsilon_1 \,, \;\; b \to \frac{\Upsilon_2 + \Upsilon_1}{2} \,; \quad \Upsilon_r \equiv \Upsilon(\rho_r) \,.$$

Let us factorize $S_{\rho_2\rho_1}(y)$ as follows

$$S_{\rho_2\rho_1}(y) \simeq \widetilde{S}_{\rho_2\rho_1}(y) \, s_{\rho_2\rho_1}(y) \,. \tag{95}$$

Then we obtain the following equation for $s_{\rho_2\rho_1}(y)$)

$$\frac{\partial s_{\rho_2\rho_1}(y)}{\partial y} = \int_{\rho_1}^{\rho_2} d\rho_0 \frac{R(\rho_0)}{T(\rho_0)} \left(s_{\rho_2\rho_0}(y) s_{\rho_0\rho_1}(y) - s_{\rho_2\rho_1}(y)\right) . \tag{96}$$

Therefore the fan equation has a renormalization group property. Namely, $s_{\rho_2\rho_1}(y)$ satisfies the same equation as $S_{\rho_2\rho_1}(y)$, but in the new variables Υ, defined by (93). As a result, we obtain the following "Backlund" transformation between different solutions of the fan equation

$$e^{(\rho_2-\rho_1)\,y} S^{(1)}_{\rho_2\rho_1}(y) = \frac{R_{12}(\rho_2)}{T_{12}(\rho_1)} \, e^{(\Upsilon_2-\Upsilon_1)\,y} \, S^{(2)}_{\Upsilon_2\Upsilon_1}(y) \tag{97}$$

for arbitrary $R(\rho)$ and $T(\rho)$, provided that the new variable satisfies

$$\Upsilon(\rho) = \int^{\rho} d\rho_0 \frac{R_{12}(\rho_0)}{T_{12}(\rho_0)} \,. \tag{98}$$

The function $\Upsilon(\rho)$ is chosen to grow with ρ and to have the additional constraint

$$\Upsilon(\pm\infty) = \pm\infty\,. \tag{99}$$

We can once more perform the above transformation:

$$e^{(\Upsilon_2-\Upsilon_1)\,y}\,S^{(2)}_{\Upsilon_2\Upsilon_1}(y) = \frac{R_{23}(\Upsilon_2)}{T_{23}(\Upsilon_1)}e^{(\Gamma_2-\Gamma_1)\,y}\,S^{(3)}_{\Gamma_2\Gamma_1}(y)\,,\ \Gamma_r = \Gamma(\Upsilon_r)\,, \tag{100}$$

where

$$\Gamma(\Upsilon) = \int^{\Upsilon} d\Upsilon_0 \frac{R_{23}(\Upsilon_0)}{T_{23}(\Upsilon_0)}\,, \tag{101}$$

and verify its group property

$$e^{(\rho_2-\rho_1)\,y}S^{(1)}_{\rho_2\rho_1}(y) = \frac{R_{13}(\rho_2)}{T_{13}(\rho_1)}\,e^{(\Gamma_2-\Gamma_1)\,y}\,S^{(3)}_{\Gamma_2\Gamma_1}(y)\,,\ \Gamma_r = \Gamma(\Upsilon(\rho_r))\,. \tag{102}$$

Here

$$\frac{R_{13}(\rho_2)}{T_{13}(\rho_1)} = \frac{R_{12}(\rho_2)}{T_{12}(\rho_1)}\,\frac{R_{23}(\Upsilon_2)}{T_{23}(\Upsilon_1)} \tag{103}$$

and

$$\Gamma(\Upsilon(\rho)) = \int^{\Upsilon(\rho)} d\Upsilon_0 \frac{R_{23}(\Upsilon_0)}{T_{23}(\Upsilon_0)} = \int^{\rho} d\rho_0 \frac{R_{12}(\rho_0)}{T_{12}(\rho_0)}\,\frac{R_{23}(\Upsilon(\rho_0))}{T_{23}(\Upsilon(\rho_0))}\,. \tag{104}$$

In particular, for

$$\Gamma(\Upsilon(\rho)) = \rho \tag{105}$$

one obtains

$$\frac{R_{12}(\rho)}{T_{12}(\rho)}\,\frac{R_{23}(\Upsilon(\rho))}{T_{23}(\Upsilon(\rho))} = 1\,. \tag{106}$$

Generally,

$$\frac{R_{13}(\rho_2)}{T_{13}(\rho_1)} = \frac{R_{12}(\rho_2)}{T_{12}(\rho_1)}\,\frac{R_{23}(\Upsilon_2)}{T_{23}(\Upsilon_1)} \neq 1\,, \tag{107}$$

and we thus obtain a family of solutions

$$S^{(f)}_{\rho_2\rho_1}(y) = \frac{f(\rho_2)}{f(\rho_1)}\,S_{\rho_2\rho_1}(y) \tag{108}$$

for arbitrary $f(\rho)$. The symmetry group of the fan equation is a product of this abelian transformation of the dipole S-matrix $S_{\rho_2\rho_1}(y)$ and of the reparametrization group (98).

Without any loss of generality it is convenient to write the action of the second group on $S_{\rho_2\rho_1}(y)$ in the form

$$e^{(\rho_2-\rho_1)\,y}S^{(a)}_{\rho_2\rho_1}(y) = a(\rho_2)\,a(\rho_1)\,e^{(\Upsilon_2-\Upsilon_1)\,y}\,S_{\Upsilon_2\Upsilon_1}(y)\,,\ \Upsilon_r = \Upsilon(\rho_r) = \int^{\rho_r} d\rho\,a^2(\rho)\,, \tag{109}$$

where $a(\rho) = \sqrt{R_{12}(\rho)/T_{12}(\rho)}$. We can express this equation directly in terms of the transformation $\rho' = \Upsilon(\rho)$

$$e^{(\rho_2-\rho_1)\,y}S^{(1)}_{\rho_2\rho_1}(y) = \sqrt{\frac{d\Upsilon(\rho_2)}{d\rho_2}\frac{d\Upsilon(\rho_1)}{d\rho_1}}\, e^{(\Upsilon_2-\Upsilon_1)\,y}\, S^{(2)}_{\Upsilon_2\Upsilon_1}(y)\,. \tag{110}$$

It means that under the general covariance transformation $\rho \to \Upsilon(\rho)$the quantity

$$T_{\rho_2\rho_1}(y) = e^{(\rho_2-\rho_1)y}\, S_{\rho_2\rho_1}(y) \tag{111}$$

transforms as a tensor with the spinor indices ρ_2, ρ_1. The functional expressing the S-matrix $S_{\rho_2\rho_1}(y)$ through initial conditions

$$S_{\rho_2\rho_1}(y) = \Theta\left(S_{\rho_2\rho_1}(0)\right) \tag{112}$$

should be invariant under the full symmetry group (together with the above multiplication of $S^{(1)}_{\rho_2\rho_1}$ by the ratio $f(\rho_2)/f(\rho_1)$). In particular we can use the above transformations to change in a convenient way the initial conditions

$$S^{(1)}_{\rho_2\rho_1}(0) = \frac{R(\rho_2)}{T(\rho_1)}\, S^{(2)}_{\Upsilon_2\Upsilon_1}(0)\,. \tag{113}$$

8 General solution of the fan equation

The fan equation for the function

$$G_{\rho_2\rho_1}(y) = e^{(\rho_2-\rho_1)y}\, S_{\rho_2\rho_1}(y) \tag{114}$$

can be written as follows

$$\frac{\partial G_{\rho_2\rho_1}(y)}{\partial y} = \int_{\rho_1}^{\rho_2} d\rho_0\, G_{\rho_2\rho_0}(y)\, G_{\rho_0\rho_1}(y)\,. \tag{115}$$

The function $G_{\rho_2\rho_1}(y)$ can be considered as a matrix element of the infinite-dimensional matrix $G(u)$ having a triangular form, in accordance with the fact that it is nonzero only for $\rho_2 \geq \rho_1$

$$G_{\rho_2\rho_1}(y) = \langle\rho_2|G|\rho_1\rangle\,. \tag{116}$$

We can define the product of two matrices $G^a(u)$ and $G^b(u)$ as the matrix $G^c(u)$ with the matrix element

$$G^c_{\rho_2\rho_1}(y) = \int_{\rho_1}^{\rho_2} d\rho_0\, G^a_{\rho_2\rho_0}(y)\, G^b_{\rho_0\rho_1}(y)\,. \tag{117}$$

Correspondingly, the inverse matrix $G^{-1}(u)$ is defined as the matrix satisfying the relations

$$\delta(\rho_2-\rho_1)=\int_{\rho_1}^{\rho_2} d\rho_0\, G_{\rho_2\rho_0}(y)\,(G^{-1})_{\rho_0\rho_1}(y)=\int_{\rho_1}^{\rho_2} d\rho_0\,(G^{-1})_{\rho_2\rho_0}(y)\,G_{\rho_0\rho_1}(y)\,. \tag{118}$$

Its matrixelements $(G^{-1})_{\rho_2\rho_1}(y)$ are non-zero only for $\rho_2 \geq \rho_1$. Thus, the operators $G(y)$ belong to the (solvable) group of triangular matrices. The matrix element of the unit element of this group is $\delta(\rho_2-\rho_1)$.

In the operator form the fan equation looks as follows

$$\frac{\partial G(y)}{\partial y}=G^2(y)\,, \tag{119}$$

and its general solution has the meromorphic property of Kowalewskaya-Painleve [24]

$$G(y)\;=\frac{G(0)}{1-y\,G(0)}\;. \tag{120}$$

Note that in accordance with the Ricatti theory the operator G^{-1} satisfies the free Newton equaton

$$\frac{d^2}{dy^2}\,G^{-1}(y)=0. \tag{121}$$

The corresponding solution for the S-matrix $S_{\rho_2\rho_1}(y)$ written in the operator form is

$$S(y)\;=e^{-\rho y}\frac{G(0)}{1-y\,G(0)}\;e^{\rho y}, \tag{122}$$

where ρ is the operator with the eigenvalues ρ_1 for the eigenfunctions $|\rho_1\rangle$. This solution is similar to the general solution of a linear equation expressed in terms of its Green function. In a matrix form it can be expanded in a series as follows

$$e^{(\rho_2-\rho_1)\,y}\,S_{\rho_2\rho_1}(y)=S_{\rho_2\rho_1}(0)+$$
$$y\int_{\rho_1}^{\rho_2} d\rho_0\,S_{\rho_2\rho_0}(0)\,S_{\rho_0\rho_1}(0)+y^2\int_{\rho_1}^{\rho_2} d\rho_{0'}\int_{\rho_1}^{\rho_{0'}} d\rho_0\,S_{\rho_2\rho_{0'}}(0)\,S_{\rho_{0'}\rho_0}(0)S_{\rho_0\rho_1}(0)+\dots\,. \tag{123}$$

In particular, according to above discussions one can simplify this general result as follows

$$\widetilde{S}_{\rho_2\rho_1}(y)=e^{(\chi(\rho_2)-\chi(\rho_1))y}\,\frac{R(\rho_2)}{T(\rho_1)}\,,\;\chi(\rho)=\int_0^{\rho} d\rho_0\left(\frac{R(\rho_0)}{T(\rho_0)}-1\right), \tag{124}$$

providing that $S_{\rho_2\rho_1}(0)$ has the factorized form

$$\widetilde{S}_{\rho_2\rho_1}(0)=\frac{R(\rho_2)}{T(\rho_1)}\,. \tag{125}$$

It is also obvious that the solutions $S_{\rho_2\rho_1}(y)$ for different $S_{\rho_2\rho_1}(0)$ are related each to other by the above established group of transformations depending on two arbitrary functions $f(\rho)$ and $\Upsilon(\rho)$

$$e^{(\rho_2-\rho_1)\,y}S^{(f,\Upsilon)}_{\rho_2\rho_1}(y)=\frac{f(\rho_2)}{f(\rho_1)}\sqrt{\frac{d\Upsilon(\rho_2)}{d\rho_2}\frac{d\Upsilon(\rho_1)}{d\rho_1}}e^{(\Upsilon_2-\Upsilon_1)\,y}\,S_{\Upsilon_2\Upsilon_1}(y)\,,\;\Upsilon_r=\Upsilon(\rho_r)\,. \tag{126}$$

In particular, we can consider the case where the initial condition $S_{\Upsilon_2\Upsilon_1}(0)$ is only a function of $\Upsilon_2-\Upsilon_1$, which gives us a possibility to construct the solution depending on three arbitrary functions

$$e^{(\rho_2-\rho_1)\,y}S^{(f,\Upsilon,s)}_{\rho_2\rho_1}(y)=\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{dp}{2\pi i}\,\frac{c^{(2)}_{\rho_2}(p)\,c^{(1)}_{\rho_1}(p)}{(s(p))^{-1}-y}\,, \tag{127}$$

where the parameter σ is chosen from the condition that all singularities of the integrand are situated to the left of the integration contour in p and

$$s(p)=\int_0^\infty d\Upsilon\,e^{-\Upsilon\,p}\,S_{\frac{\Upsilon}{2},-\frac{\Upsilon}{2}}(0)\,,$$

$$c^{(2)}_{\rho_2}(p)=\sqrt{\frac{d\Upsilon(\rho_2)}{d\rho_2}}\,e^{\Upsilon_2\,p}\,f(\rho_2)\,,\;c^{(1)}_{\rho_1}(p)=\sqrt{\frac{d\Upsilon(\rho_1)}{d\rho_1}}\,\left(e^{\Upsilon_1\,p}f(\rho_1)\right)^{-1}\,. \tag{128}$$

The solution $S^{(f,\Upsilon,s)}$ can be considered as a particular case of the general solution transformed to a diagonal form by an appropriate similarity transformation. The function $s(p)$ has the interpretation of the eigenvalue of $S_{\rho_2\rho_1}(0)$, and $c^{(2)}_{\rho_1}(p)$ is its eigenfunction. The similarity transformation is unitary ($c^{(2)}=c^{(1)+}=1/c^{(1)}$) provided that $f(\rho)$ is a phase: $f^*=f^{-1}$.

In general the functions $c^{(r)}_\rho(p)$ have a more complicated (non-exponential) dependence on p which enumerates the eigenvalues. This dependence can be found from the solution of the equation

$$\int_{-\infty}^{\infty}d\,\rho_1\,S_{\rho_2\rho_1}(0)c_{\rho_1}(p)=s(p)\,c_{\rho_2}(p)\,. \tag{129}$$

For the triangular matrix $S_{\rho_2\rho_1}(0)$ the eigenfunction $c_\rho(p)$ can be calculated easily, provided that we substitute the continuous coordinate ρ by a finite number of points ρ_k. In general, the behaviour of $S_{\rho_2\rho_1}(y)$ at large y depends on the initial conditions encoded in the asymptotics of the functions $c_\rho(p)$ at large p. Due to the colour transparancy property we have

$$S_{\rho_2\rho_1}(0)<1\,,\;\lim_{\rho_2\to\rho_1}S_{\rho_2\rho_1}(0)\;=1-c\rho_{21}^\gamma,\;\;\gamma>0\;, \tag{130}$$

where the coefficient $c>0$ is a function of b. At large y and fixed ρ_r it is natural to expect an universal dependence of $S_{\rho_2\rho_1}(y)$ from y. In particular, for large c the S-matrix should

have the form of a smeared $\delta(\rho_{21})$-function. It is related to the fact that in this case, as a result of the y-evolution, the average size of the dipole tends to zero, and one can neglect the b-dependence of S (up to a common factor). In the pre-asymptotics a scaling behaviour of $S_{\rho_2\rho_1}(y)$ is possible (cf. section 4).

It is important that the solution of the Balitsky-Kovchegov equation satisfies also the linear integral equation

$$G_{\rho_2\rho_1}(y) = G_{\rho_2\rho_1}(0) + y \int_{\rho_1}^{\rho_2} d\,\rho \; G_{\rho_2\rho_0}(0)\, G_{\rho_0\rho_1}(y)\,. \tag{131}$$

In particular, the theory of the Fredholm equations can be applied to this equation. In this case, the perturbation theory in y should work rather well. For large y one can use a quasi-classical approximation. We hope to return to the investigation of the this linear equation in future publications.

9 Conclusions

In this paper we have obtained analytic solutions of a simplified version of the nonlinear BK equation, namely the BK equation in one transverse dimension. Our explicit formulae allow to study, for any initial condition, the rapidity evolution, both as a function of the dipole size and of the dipole position in impact parameter.

We also have found several remarkable properties of the nonlinear equation: the general solution has the meromorphic Kowalewskaya-Painlevé property, and the equation has symmetries which are reminiscent of general covariance. Some of these results might be helpful in addressing also the more realistic case of two transverse dimensions.

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Appendix

Here we derive the reccurence relations for the coefficients a_n in (83). Inserting the expansion (83) in (80) one can obtain the relation

$$(n+1)\frac{1}{\epsilon^2}\sum_{n=0}^{\infty}\frac{a_n}{(\epsilon b)^n}=\frac{1}{\epsilon^2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{a_k}{(\epsilon b)^k}\frac{a_l}{(\epsilon b)^l}\left((1+\frac{u}{\epsilon b}\frac{d}{dx})^{-k}\frac{1}{x^{l+1}}\right)|_{x=1}\left((1-\frac{u}{\epsilon b}\frac{d}{dx})^{-l}\frac{1}{x^{k+1}}\right)|_{x=1}\,.$$

Using the expansion

$$\left(\frac{1}{(1+z\frac{d}{dx})^k}\frac{1}{x^{l+1}}\right)|_{x=1}=\sum_{r=0}^{\infty}\frac{\Gamma(k+r)}{\Gamma(r+1)\Gamma(k)}\left(\left(-z\frac{d}{dx}\right)^r\frac{1}{x^{l+1}}\right)|_{x=1}=\sum_{r=0}^{\infty}C^r_{k,l+1}z^r\ ,$$

where

$$C^r_{k,l}=\frac{\Gamma(k+r)\Gamma(l+r)}{\Gamma(r+1)\Gamma(k)\Gamma(l)}\ ,$$

one can easily verify that $a_0=1$, and a_1 is arbitrary. For the coefficients a_n with $n>1$ we obtain the recurrence relation

$$(n-1)a_n=$$

$$\sum_{k=1}^{n-1}a_k\,C^{n-k}_{k,1}\left(u^{n-k}+(-u)^{n-k}\right)+\sum_{m=2}^{n}\sum_{k=1}^{m-1}a_k\;a_{m-k}\,u^{n-m}\;\sum_{l=0}^{n-m}(-1)^l\,C^l_{k,m-k+1}\;C^{n-m-l}_{m-k,k+1}\ .$$

Evidently, the terms with odd powers of u are cancelled.

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